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# Cumulants of the three-state Potts model and of nonequilibrium models with $C_{3v}$ symmetry

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Received 2 January 2002, in final form 23 April 2002

Published 21 June 2002

Online at [stacks.iop.org/JPhysA/35/5379](http://stacks.iop.org/JPhysA/35/5379)

## Abstract

The critical behaviour of two-dimensional stochastic lattice gas models with  $C_{3v}$  symmetry is analysed. We study the cumulants of the order parameter for the three-state (equilibrium) Potts model and for two irreversible models whose dynamic rules are invariant under the symmetry operations of the point group  $C_{3v}$ . By means of extensive numerical analysis of the phase transition we show that irreversibility does not affect the critical behaviour of the systems. In particular, we find that the Binder reduced fourth-order cumulant takes a universal value  $U^*$  which is the same for the three-state Potts model and for the irreversible models. The same universal behaviour is observed for the reduced third-order cumulant.

PACS numbers: 05.50.+q, 05.70.Lu

## 1. Introduction

The critical behaviour of nonequilibrium systems has been amply studied in the last few years [1–4]. These studies consider stochastic lattice models and probabilistic cellular automata that evolve in time according to an irreversible dynamics, that is, a dynamics that lacks detailed balance. An important aspect to be considered is the role of symmetry. Distinct systems with the same symmetry are expected to have similar critical behaviour. The symmetry is to be found in the Hamiltonian for reversible systems and in the dynamics for the irreversible systems. Among the irreversible models, there are models that have a reversible counterpart with the same symmetries. In this context, it has been established by the following statement: models with *up-down* symmetry, similar to the Ising model, and defined on the same lattice, reversible or not, are in the same universality class [5]. This has been verified numerically for a large number of models [6–12]. We note that the same universal behaviour is also observed for dynamic phase transitions in Ising models in oscillating fields [13–15].

Recently [16–18], we have argued that this statement can be extended to models with other symmetries. In fact, in these works, we considered probabilistic cellular automata with dynamics that possess  $C_{3v}$  symmetry and verified that the values of the critical exponents, both static and dynamic, for irreversible systems are the same as those of the *equilibrium* three-state Potts. That is, irreversibility is irrelevant regarding the values of the critical exponents in systems with the symmetry of the Potts models. In this paper, we complete these analyses by performing a systematic study of the cumulants at the critical point. We consider stochastic lattice gas models with  $C_{3v}$  symmetries: the three-state Potts model [19] and two irreversible models, and focus our attention on the determination of the third- and fourth-order cumulants of the order parameter. We present a Monte Carlo study of these properties for models defined in a regular square lattice. Our results show that the values of the cumulants, at the critical point, obtained for the present nonequilibrium models, and those associated with the two-dimensional three-state Potts (equilibrium) model are the same within numerical errors.

The paper is organized as follows. In section 2 we present the models to be studied. In section 3, the quantities of interest in the study of the phase transition and their scaling properties are defined. The values of the cumulants in the limit of infinite temperature and zero temperature are discussed in section 4. Sections 5 and 6 show numerical calculations and concluding remarks.

## 2. Models

Consider a regular lattice of  $N$  sites in which each site can be in one of three states. At each site we attach a stochastic variable  $\sigma_i$  that takes the values 1, 2 and 3. The state of the system can be represented by  $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_N)$ . The time evolution equation for  $P(\sigma, t)$ , the probability of state  $\sigma$  at time  $t$ , is given by the master equation

$$\frac{d}{dt}P(\sigma, t) = \sum_{\sigma'} \{W(\sigma | \sigma')P(\sigma', t) - W(\sigma' | \sigma)P(\sigma, t)\} \quad (1)$$

where the sum is over the  $3^N$  configurations of the system.  $W(\sigma | \sigma')$  is the transition rate from a state  $\sigma'$  to a state  $\sigma$ , given that at the previous time step the system was in state  $\sigma'$ . We will consider dynamics where  $\sigma$  and  $\sigma'$  can differ only by one site which we call a one-site dynamics. In this case, we have

$$\frac{d}{dt}P(\sigma, t) = \sum_{\alpha} \sum_{i=1}^N \{w_i(\sigma^{i\alpha})P(\sigma^{i\alpha}, t) - w_i(\sigma)P(\sigma, t)\} \quad (2)$$

where  $w_i(\sigma)$  is the transition probability per site and

$$\sigma^{i\alpha} = (\sigma_1, \sigma_2, \dots, \sigma_i + \alpha, \dots, \sigma_N) \quad (3)$$

with  $\alpha = 1, 2$ . The states are defined modulo 3.

The transition probability  $W(\sigma | \sigma')$  is invariant under certain symmetry operations, that is,  $W(R\sigma | R\sigma') = W(\sigma | \sigma')$  where  $R$  is a symmetry operation. For the present models the symmetry operations  $R$  are those that act on all sites transforming each of them in the same manner. One of the symmetry operations is the rotation operation  $1 \rightarrow 2, 2 \rightarrow 3$  and  $3 \rightarrow 1$ . Another is the operator  $2 \rightleftharpoons 3$  with state 1 fixed. If the three states are placed on the vertices of an equilateral triangle they correspond, respectively, to a rotation operation by  $120^\circ$  and a specular operation. These symmetry operations then define the point group  $C_{3v}$ .

### 2.1. Equilibrium model and Metropolis prescription

The Hamiltonian of the three-state Potts model is

$$H = \sum_{(ij)} -J\delta(\sigma_i, \sigma_j) \quad (4)$$

where  $\sigma_i = 1, 2, 3$ ,  $J > 0$  is the interacting strength between the nearest-neighbour sites  $i$  and  $j$  and  $\delta$  is the Kronecker delta.

To simulate the model we associate with it a dynamics. We consider a one-site dynamics as described by the master equation (2). In the case the model to be analysed is an equilibrium model it is necessary to use a transition probability  $w_i(\sigma)$  that satisfies a detailed balance condition. That is, in the stationary state we must have

$$w_i(\sigma^{i\alpha})P(\sigma^{i\alpha}, t) = w_i(\sigma)P(\sigma, t). \quad (5)$$

This dynamics can be defined by using the Metropolis prescription. We choose a site  $i$  and a state  $\alpha$  and calculate

$$w_i(\sigma) = \min\{1, \exp(-\beta\Delta H)\} \quad (6)$$

where  $\Delta H = H(\sigma) - H(\sigma^{i\alpha})$  is the difference between the energy of the state  $\sigma$  and the energy of the state  $\sigma^{i\alpha}$ . The parameter  $\beta$  is associated with the inverse of the temperature  $T$ . Numerically, we studied the critical point associated with the model by performing Monte Carlo simulations. This procedure is described in section 6.

### 2.2. Nonequilibrium models

The nonequilibrium models are defined as follows. For the case of a square lattice we denote the transition probability  $w_0(\sigma)$  by  $w(\sigma_0|\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ , where the sites 1, 2, 3 and 4 are the first neighbours of site 0.

#### 2.2.1. Symmetric stochastic lattice gas model

- (a) If in the neighbourhood of a given site there is a majority of sites which are in one state then, independently of the state of the site, it changes to the state of the majority with probability  $p$ . It changes to one of the two other states with probability  $(1 - p)/2$ .
- (b) If no state is in majority then the site assumes either state with equal probability.

According to the local rules of the model we have

$$w(1 | 1111) = w(1 | 1112) = w(1 | 1113) = w(1 | 1123) = p$$

and

$$w(1 | 1122) = w(1 | 1133) = w(1 | 2233) = 1/3. \quad (7)$$

The other rules are obtained by permutation of the neighbouring sites and by cyclic permutation of the states.

**2.2.2. Majority stochastic lattice gas model.** The model consists of a stochastic lattice gas model where the site transition probabilities follow rules similar to those of the majority vote model [8]. The chosen site adopts the same value as that of the majority of the nearest-neighbour sites with probability  $p$ . It adopts the state of the other states with probability  $q/2 = (1 - p)/2$ . If there is an equal number of nearest-neighbour sites in the same state

then the chosen site adopts each state with probability  $p/2$  or it assumes the other state with probability  $q$ . That is,

$$\begin{aligned} w(1 | 1111) &= w(1 | 1112) = w(1 | 1113) = w(1 | 1123) = p \\ w(1 | 1122) &= w(1 | 1133) = p/2 \\ w(1 | 2233) &= (1 - p) \\ w(1 | 1222) &= w(1 | 1333) = w(1 | 3222) = w(1 | 3312) = (1 - p)/2. \end{aligned} \quad (8)$$

The other rules are obtained by permutation of the neighbouring sites and by cyclic permutation of the states.

It is straightforward to check that the transition probabilities  $w_i(\sigma)$ , for both models, are invariant under the symmetry operations of the group  $C_{3v}$ .

The nonequilibrium models have the same symmetries as the Hamiltonian of the three-state Potts model, given in (4), although in the present case the models are not defined by a Hamiltonian and do not satisfy detailed balance condition (5). That is, these models are microscopically irreversible.

### 3. Cumulants and scaling properties

A convenient way of analysing the present models is through the use of the variables

$$x_\alpha = \frac{1}{N} \sum_{i=1}^N \left( \delta(\sigma_i, \alpha) - \frac{1}{3} \right) \quad (9)$$

where  $\alpha$  assumes the values 1, 2 and 3 and  $\delta(x, y)$  is the Kronecker delta. The order parameter has three components  $x_1, x_2$  and  $x_3$ , but just two of them are independent and the following property

$$x_1 + x_2 + x_3 = 0$$

holds.

It is useful also to introduce a set of homogeneous functions  $I_n(x_1, x_2, x_3)$ , of a given order  $n$ , which are invariant under the symmetry operations  $R$  defined above. There is just one independent second-order invariant given by

$$I_2 = \frac{1}{3}(x_1^2 + x_2^2 + x_3^2) \quad (10)$$

and just one independent third-order invariant

$$I_3 = -\frac{2}{9}(x_1^3 + x_2^3 + x_3^3). \quad (11)$$

The fourth-order invariant function is

$$I_4 = \frac{2}{9}(x_1^4 + x_2^4 + x_3^4). \quad (12)$$

Again there is just one independent fourth-order invariant.

#### 3.1. Cumulants

In the present study, the quantities of interest are the order parameter,

$$m = |\vec{m}| = \langle \sqrt{I_2} \rangle \quad (13)$$

and the reduced cumulant,

$$U_{24} = \frac{\langle I_4 \rangle}{\langle I_2 \rangle^2} \quad (14)$$

and, also, the Binder fourth-order cumulant which in terms of  $I_4$  and  $I_2$  defined above is given by

$$U = 1 - \frac{1}{3}U_{24}. \tag{15}$$

We also analysed the behaviour of the order three reduced cumulant,

$$U_{23} = \frac{\langle I_3 \rangle}{\langle I_2 \rangle^{3/2}}. \tag{16}$$

### 3.2. Scaling properties

The order parameter  $\vec{m}$  has two independent components  $x$  and  $y$  and we will denote

$$\vec{m} = \frac{1}{\sqrt{2}}(x \vec{i} + y \vec{j}) \tag{17}$$

where  $x$  and  $y$  are related to  $x_1, x_2$  and  $x_3$  by the relations

$$x_1 = -\frac{\sqrt{3}}{2}x - \frac{1}{2}y \quad x_2 = \frac{\sqrt{3}}{2}x - \frac{1}{2}y \tag{18}$$

$$x_3 = y. \tag{19}$$

The invariants (10), (11) and (12) are written, in terms of  $x$  and  $y$ , as

$$I_2 = \frac{1}{2}(x^2 + y^2) \tag{20}$$

$$I_3 = \frac{1}{2}\left(x^2y - \frac{y^3}{3}\right) \tag{21}$$

$$I_4 = \frac{1}{4}(x^4 + 2x^2y^2 + y^4). \tag{22}$$

Our main interest is to calculate the moments of the distribution associated with the order parameter. The moment  $M_n$  of order  $n$  can be defined by

$$M_n = \langle |m|^n \rangle = \int |m|^n P(m, \epsilon, L) dm$$

where  $P(m, \epsilon, L)$  is the probability distribution of  $m = |\vec{m}|$ , where  $\vec{m} = \frac{1}{\sqrt{2}}(x, y)$ , and  $\epsilon$  is equal to the deviation of the external parameter from its critical value and  $L$  is the system size. We assume that

$$P(m, \epsilon, L) = L^{\beta/\nu} \phi(m/\epsilon^\beta, L\epsilon^\nu). \tag{23}$$

Defining  $z = m/\epsilon^\beta$ ,

$$M_n = \epsilon^{\beta(n+1)} L^{\beta/\nu} \int |z|^n \phi(z, L\epsilon^\nu) dz.$$

Then,

$$M_n = \epsilon^{\beta n} (\epsilon^\nu L)^{\beta/\nu} F_n(\epsilon L^{1/\nu}) \tag{24}$$

where  $F_n(Y)$  is an universal function.

From relation (24), we get the following scaling forms

$$M_1 = \langle m \rangle = L^{-\beta/\nu} \tilde{m}(\epsilon L^{1/\nu}) \tag{25}$$

$$\frac{M_4}{(M_2)^2} = \tilde{U}_{24}(\epsilon L^{1/\nu}) \tag{26}$$

and

$$\frac{M_3}{(M_2)^{3/2}} = \tilde{U}_{23}(\epsilon L^{1/\nu}) \quad (27)$$

where  $\epsilon$  is the deviation of the external parameter from its critical value and  $\tilde{m}(x)$ ,  $\tilde{U}_{23}(x)$  and  $\tilde{U}_{24}(x)$  are universal functions.

For an infinite system the correlation length diverges as  $\xi \sim \epsilon^{-\nu}$  and the scaling forms give the behaviour  $m \sim \epsilon^\beta$  for the order parameter. Moreover, the reduced cumulants  $U_{24}$  and  $U_{23}$ , defined in equations (14) and (16), are expected to attain, according to (26) and (27), a universal value at the critical point, which does not depend on the lattice size. The same behaviour, of course, must hold for the reduced fourth-order cumulant  $U$  given in equation (15).

#### 4. Exact results

When the temperature  $T \rightarrow 0$  the equilibrium model defined by the Hamiltonian (4) will be in the ordered phase and with probability 1 in one of the three Potts states. For the nonequilibrium models, defined in (7) and (8), this limit corresponds to  $p \rightarrow 1$ .

In this limit the following behaviour is expected,

$$\langle (x_\alpha)^n \rangle \rightarrow \left\{ \left(\frac{1}{3}\right) \left(\frac{2}{3}\right)^n + \left(\frac{1}{3}\right) \left(-\frac{1}{3}\right)^n + \left(\frac{1}{3}\right) \left(-\frac{1}{3}\right)^n \right\} \quad (28)$$

with  $\alpha = 1, 2$  and  $3$  and  $n = 1, 2, 3, \dots$ . Depending on the initial conditions the system will be, for all sites in the lattice, in the states where  $\sigma_i = 1$ ,  $\sigma_i = 2$  or  $\sigma_i = 3$ . The factor  $\left(\frac{1}{3}\right)$  in equation (28) takes into account this fact. For example, the second-order invariant, defined in expression (10), will attain the value

$$I_2 \rightarrow \frac{1}{3} \left\{ \left(\frac{1}{3}\right) \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right) \left(-\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right) \left(-\frac{1}{3}\right)^2 \right\}$$

that is,  $I_2 \rightarrow 2/9$  when  $T \rightarrow 0$  ( $p \rightarrow 1$ ).

Following the same procedure, the limiting values of the third- and fourth-order invariants and the reduced cumulants can be easily evaluated. In particular, the values of the fourth-order cumulant, defined in (14), in the limit  $T \rightarrow 0$  ( $p \rightarrow 1$ ), will be

$$U_{24} \rightarrow 1 \quad (29)$$

which implies that the Binder fourth-order cumulant, according to equation (15), takes the limit

$$U \rightarrow \frac{2}{3}. \quad (30)$$

On the other hand, when the temperature  $T \rightarrow \infty$  ( $p \rightarrow 1/3$ ) the equilibrium system (the nonequilibrium symmetric model) is in the disordered state,

$$x_1 = x_2 = x_3 = 0.$$

So the probability distribution associated with the order parameter is a distribution of independent variables, a Gaussian distribution, and we can write

$$P(x, y) = \frac{1}{2\pi a} \exp[-(x^2 + y^2)/2a] \quad (31)$$

with

$$a = \langle (x^2 + y^2)/2 \rangle.$$

So in this limit we have that expressions (20) and (22) are related by

$$I_4 = 3(I_2)^2 \quad (32)$$

which implies that

$$U_{24} \rightarrow 3 \quad (33)$$

and

$$U \rightarrow 0. \quad (34)$$

The third-order invariant  $I_3 \rightarrow 0$  when  $T \rightarrow \infty$  ( $p \rightarrow 1/3$ ). So  $U_{23} \rightarrow 0$ .

## 5. Monte Carlo simulations

The system evolves in time according to the local rules and eventually reaches a steady state that can be of two types: a disordered steady state, where there is an equal average number of sites in each one of the three Potts states; or an ordered steady state characterized by the predominance of sites in one of the Potts states.

The simulation of the equilibrium and the nonequilibrium models with  $C_{3v}$  symmetry was performed by considering square lattices with  $L^2 = N$  sites, and periodic boundary conditions. Each simulation started with a configuration generated at random and averages over several simulations were taken to get the final results.

### 5.1. Equilibrium model

We consider several values of the external parameter, the temperature  $T$ . We pick a site  $i$  at random and then apply the Metropolis prescription to update site  $i$  according to the expression (6) as follows. Let the state of the site  $i$  be  $\sigma_i$ . We change the site variable to  $\sigma_i + \alpha$  and calculate  $\Delta H$  according to the expression (4) considering the nearest-neighbour sites of the site  $i$  (which have not changed, since we are considering a one-site dynamics). If  $\Delta H \leq 0$ , then the new state will be  $\sigma^{i\alpha} = (\sigma_1, \dots, \sigma_{i+\alpha}, \dots, \sigma_N)$ . Otherwise, if  $\Delta H > 0$ , we calculate  $p = \exp(-\beta\Delta H)$  and generate a random number  $\zeta$  equally distributed in the interval  $[0, 1]$ . If  $\zeta \leq p$  then the new state will be  $\sigma^i$ , otherwise the state does not change, that is, remains  $\sigma$ .

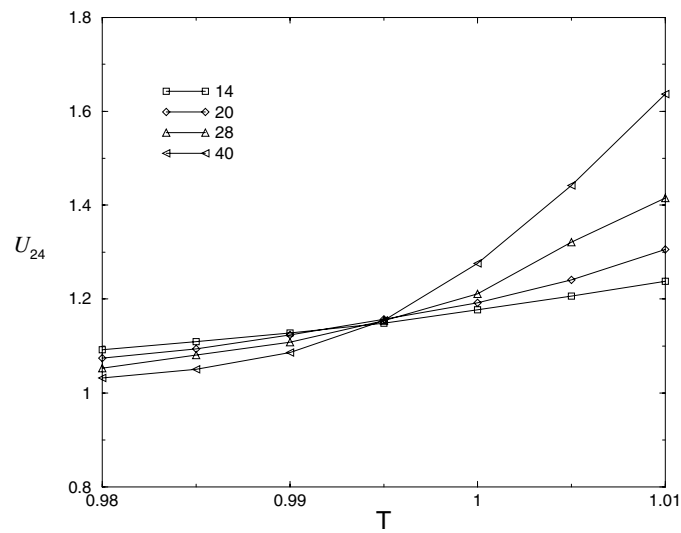
The system evolves in time until it reaches a stationary state. The time taken by the system to reach the stationary state depends on the temperature and the lattice size. After discarding the first configurations, we used the following states in order to evaluate the state function cumulants of the distribution probability associated with the order parameter. The stationary states are equilibrium stationary states, i.e. they satisfy the detailed balance condition. As expected, we found two types of stationary states: a ordered one where the order parameter  $m$  is different from zero and a disordered one where  $m = 0$ .

The critical temperature for the two-dimensional three-state Potts model is given by  $k_B T_c = 1/(\ln(\sqrt{3} + 1)) \simeq 0.99497$  [19]. In our simulation we take  $k_B = 1$  and analyse the behaviour of the cumulants (14) and (16) as a function of the temperature and for different lattice sizes. As we can see in figures 1 and 2 when  $T \rightarrow T_c$  the cumulant  $U_{24} \rightarrow U_{24}^*$  and the cumulant  $U_{23} \rightarrow U_{23}^*$ , respectively, and it follows that  $U_{24}^*$  and  $U_{23}^*$  are universal. That is, at the critical point  $U_{24}$  and  $U_{23}$  attain universal values that do not depend on the lattice size. We used the finite size scaling relations (27) and (26). Expression (27) is related to the reduced Binder [20] fourth-order cumulant that so assumes an universal value  $U^*$ . The values for these functions at the critical point were found to be

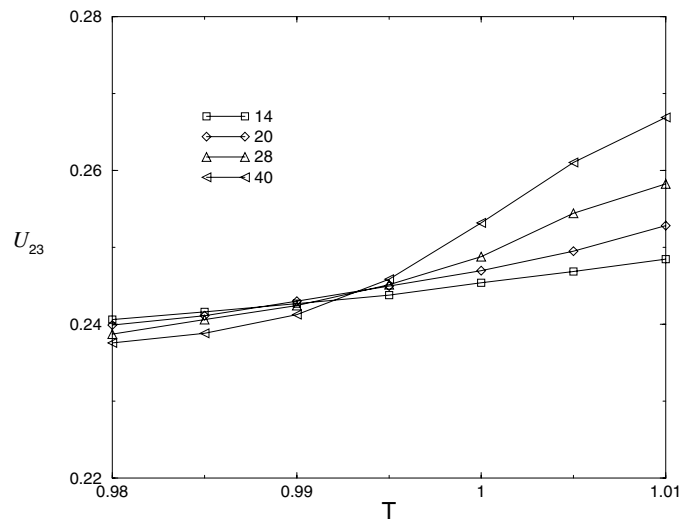
$$U_{24}^* = 1.16 \pm 0.01 \quad (35)$$

and  $U^* \simeq 0.61$ . The value of  $U_{23}$  at the critical point (see figure 2) is  $U_{23}^* = 0.245 \pm 0.01$ .





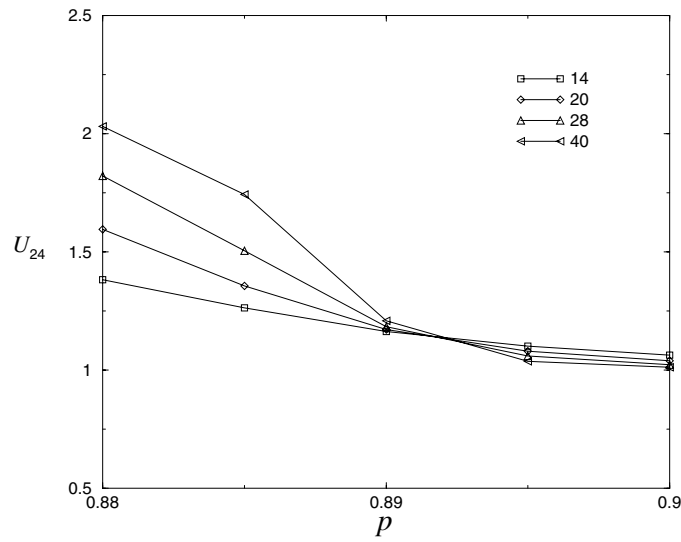
**Figure 1.** The reduced cumulant  $U_{24}$  as a function of  $T$  for  $L = 14, 20, 28$  and  $40$  (square lattices) for the three-state (equilibrium) Potts model.



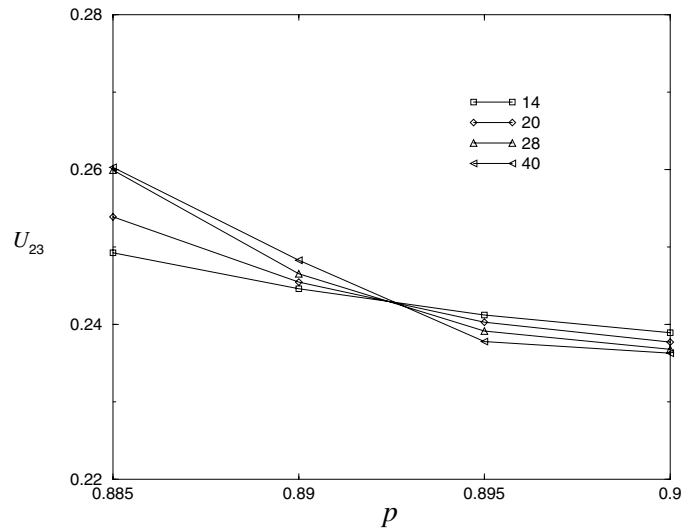
**Figure 2.** The reduced cumulant  $U_{23}$  as a function of  $T$  for  $L = 14, 20, 28$  and  $40$  (square lattices) for the three-state (equilibrium) Potts model.

### 5.2. Nonequilibrium models

We consider several values of the parameter  $p$ . At each time step just one site is chosen at random and it is updated according to the prescriptions given in section 2.2 (rules (7) for the symmetric model and rules (8) for the majority model). After a transient, which depends on the model, the size of the system and the value of  $p$ , the system attains a steady state. Our simulations show that both models exhibit continuous phase transitions with the ordered steady state ( $m \neq 0$ ) occurring at high values of  $p$ . As  $p$  is decreased the transition takes place



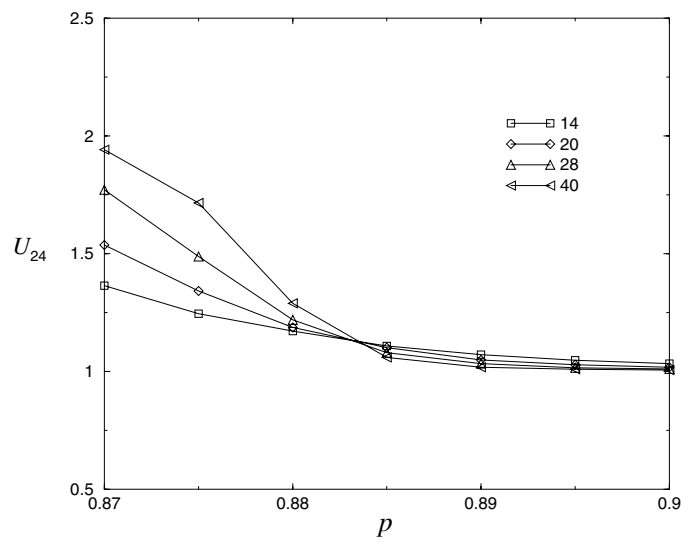
**Figure 3.** The reduced cumulant  $U_{24}$  as a function of  $p$  for  $L = 14, 20, 28$  and  $40$  (square lattices) for the three-state symmetric model (nonequilibrium).



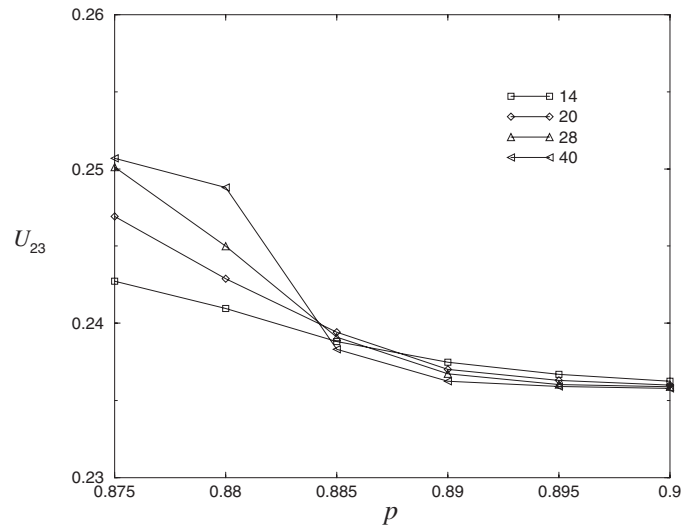
**Figure 4.** The reduced cumulant  $U_{23}$  as a function of  $p$  for  $L = 14, 20, 28$  and  $40$  (square lattices) for the symmetric model (nonequilibrium).

at a critical value  $p_c$ , which is different for each model, and the system becomes disordered ( $m = 0$ ) for  $p$  less than  $p_c$ .

Using the finite size scaling relations (27) and (26) we obtain the critical value  $p_c$  for each model. For the *symmetric lattice gas model*, as shown in figures 3 and 4, the curves of  $U_{24}$  versus  $p$  and the curves of  $U_{23}$  versus  $p$ , for different values of  $L$ , intercept at the critical point  $p_c$  estimated to be  $p_c = 0.892 \pm 0.003$ . It is worth calling to attention that this model is similar to that considered by one of us in a previous work [17]. Both models evolve in time according to the same local Markovian rules. However, the model considered in [17] is a probabilistic cellular automaton (synchronous update) whereas the present model evolves



**Figure 5.** The reduced cumulant  $U_{24}$  as a function of  $p$  for  $L = 14, 20, 28$  and  $40$  (square lattices) for the three-state (nonequilibrium) majority model.



**Figure 6.** The reduced cumulant  $U_{23}$  as a function of  $p$  for  $L = 14, 20, 28$  and  $40$  (square lattices) for the three-state (nonequilibrium) majority model.

in time according to a sequential dynamics (asynchronous update). As irreversible models are defined by the dynamics itself we do not have to expect the same value of the critical parameter. In fact they are different. We also observe that the results obtained previously [17] for the fourth-order cumulant for the probabilistic cellular automaton are not sufficient precise due to the presence of large fluctuations. In contrast, in the study of the stochastic lattice gas models considered here, we verified that the behaviour of the cumulants, both of third and fourth orders, is smooth.

Figures 5 and 6 show the curves of  $U_{24}$  versus  $p$  and the curves of  $U_{23}$  versus  $p$  for different values of  $L$  for the *majority* stochastic lattice gas model. The interception of these

curves yields the critical point estimated to be  $p_c = 0.883 \pm 0.001$ . The values attained by  $U_{23}$  and  $U_{24}$  are universal and are  $U_{23}^* = 0.244 \pm 0.01$  and  $U_{24}^* = 1.16 \pm 0.01$ . And the Binder reduced fourth-order cumulant is  $U^* \cong 0.61$ . These universal values are in agreement with the results for the (equilibrium) three-state Potts model.

## 6. Summary

We have considered systems that undergo a phase transition from a state with high symmetry to a state with lower symmetry. The phase with high symmetry is invariant under the symmetry operations of the symmetry group  $C_{3v}$ . We analysed first the equilibrium three-state Potts model, and then two irreversible models. All models have a continuous time evolution, governed by a master equation. In the first case, the model is defined by an Hamiltonian whereas in the second case they are defined only by the transition rates that do not obey the detailed balance condition. The phase transition that takes place, as an external parameter is varied, is a continuous phase transition from a disordered steady state to an ordered steady state. For the case of the equilibrium model, the phase transition occurs when the temperature  $T$  is varied and for the nonequilibrium cases when the parameter  $p$  is varied. We introduced a set of homogeneous functions that are invariant under the symmetry operations of the group  $C_{3v}$ . From these functions we define the order parameter and the cumulants. The critical points were estimated by numerical simulations on regular square lattices of different sizes and by using finite size scaling theory. Analysing the cumulants we conclude that irreversibility plays an irrelevant role in the critical behaviour and is not a property that might change the universal behaviour. In fact, the fourth- and third-order cumulants attain universal values at the critical point and we found that these values are the same for the equilibrium and the nonequilibrium systems, whenever periodic boundary conditions are considered. It is worthwhile observing that the value of the cumulants may depend on the boundary conditions, as was established for the equilibrium two-dimensional models [21].

## Acknowledgment

This work was partially supported by the Fundação de Amparo à Pesquisa do Estado de São Paulo (FAPESP) through the 01-09590-8 project.

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